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The q-Symplectic Form on the Quantum Hyperplane and Noncommutative Hamiltonian Mechanics

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The q-fields, q-curves, and q-symplectic forms on the quantum hyperplane are given by the use of the q-sequences method. With these structures we discuss a possible noncommutative Hamiltonian mechanical system and give two concrete examples.

1. INTRODUCTION AND PRELIMINARY REMARKS

In recent years, an interesting problem in theoretical physics has been how to use quantum groups and the methods of noncommutative geometry to study possible q-deformations of physical spaces and fields. However, difficulties remain, e.g., lack movable coordinates as in ordinary classical analysis, etc. For this reason, in Zhong (1993, 1994a,b, 1995a,b) I have suggested a new method, a theory of q-sequences, q-analytic functions, and noncommutative analysis on a quantum hyperplane. This method has been used to study the complete algebraization problem in the implicate order theory of Bohm (Zhong, 1995a), quantum group gauge fields and their invariants (Zhong, 1993, 1994a), nonlinear realizations of the quantum groups (Zhong, 1994b), and the associated Riemann–Hilbert problem (Zhong, 1995b). In this paper, the method is used in classical Newtonian mechanics, i.e., a noncommutative Hamiltonian mechanical system is given. For this reason, we must first define the vector fields, the form fields, the curves (the orbits of particles in a noncommutative phase space), and the symplectic form on a quantum hyperspace.

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The relations between quantum group theory and classical mechanics have been considered for some single-particle cases (e.g., Aref'eva and Volovich, 1991; Caban *et al.*, 1994). In this paper a more general method in view of the q-sequences theory is given.

For the sake of simplicity, in this paper we only consider the two-dimensional case, which corresponds to the one-dimensional motion of single particles; the corresponding quantum group is $GL_q(2)$. However, the results can be extended to the high (even)-dimensional case, which will be discussed elsewhere. According to the scheme of Manin (1988), the so-called "coordinates" of a quantum hyperplane $\mathscr{C}_2$ are defined as the generator set (x, y) of a unital associative algebra on a field (in this paper it is the real field R^1), with the commutation relation

$$xy = qyx \tag{1.1}$$

where the real deformation parameter is $q \in (0, 1]$. On the quantum hyperplane $\mathscr{C}_2$ the covariant differential calculus is (Wess and Zumino, 1990)

$$d = dx\, \partial_x + dy\, \partial_y, \qquad d^2 = 0 \tag{1.2a}$$

$$d(fg) = (df)g + f(dg) \tag{1.2b}$$

$$d(\tilde{\psi} \wedge \tilde{\varphi}) = (d\tilde{\psi}) \wedge \tilde{\varphi} + (-1)^k \tilde{\psi} \wedge d\tilde{\varphi}$$

where $\tilde{\psi}$ is a k-form. The commutation relations are

$$dx \wedge dy = -\frac{1}{q}\, dy \wedge dx \tag{1.3a}$$

$$x\, dx = q^2 dx \cdot x, \qquad x\, dy = q\, dy \cdot x + (q^2 - 1)\, dx \cdot y \tag{1.3b}$$

$$y\, dx = q\, dx \cdot y, \qquad y\, dy = q^2\, dy \cdot y$$

$$\partial_x x = 1 + q^2 x\, \partial_x + (q^2 - 1) y\, \partial_y, \qquad \partial_x y = qy\, \partial_x \tag{1.3c}$$

$$\partial_y x = qx\, \partial_y, \qquad \partial_y y = 1 + q^2 y\, \partial_y$$

$$\partial_x\, dx = \frac{1}{q^2}\, dx\, \partial_x, \qquad \partial_x dy = \frac{1}{q}\, dy\, \partial_x \tag{1.3d}$$

$$\partial_y dx = \frac{1}{q}\, dx\, \partial_y, \qquad \partial_y dy = \frac{1}{q^2}\, dy\, \partial_y + (q^2 - 1) dx\, \partial_x$$

$$\partial_x \partial_y = \frac{1}{q}\, \partial_y \partial_x \tag{1.3e}$$

The concrete actions of the quantum partial derivative operators are

$$\partial_x(x^m y^n) = q^{2n} [m]_q x^{m-1} y^n \tag{1.4}$$

$$\partial_y(x^m y^n) = q^m [n]_q x^m y^{n-1}$$

where m and n are nonnegative integers and the q-number is given by $[m]_q = (q^{2m} - 1)/(q^2 - 1) = q^{2m-2} + q^{2m-4} + \cdots + 1$.

A matrix in the quantum group $GL_q(2)$ is given by

$$M = (M^i_j) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad (i, j = 1, 2)$$

$$ab = qba, \qquad ac = qca, \qquad ad - da = \left(q - \frac{1}{q}\right)bc \tag{1.5}$$

$$bc = cb, \qquad bd = qdb, \qquad cd = qdc$$

The matrix M satisfies the Yang–Baxter relation

$$M_1 m_2 \check{R}_{12} = \check{R}_{12} M_1 M_2 \tag{1.6}$$

where corresponding to $GL_q(2)$, the Yang–Baxter matrix $\check{R}$ is

$$\check{R} = (\check{R}^{ij}_{kl}) = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q - 1/q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \qquad (i, j, k, l = 1, 2) \tag{1.7}$$

From equation (1.6), all commutation relations in (1.3) are covariant under the following transformations ($x^1 = x$, $x^2 = y$, $\partial_i = \partial_{x^i}$):

$$x^i \to x'^i = \sum_{k=1}^{2} M^i_k x^k \tag{1.8a}$$

$$dx^i \to dx'^i = \sum_{k=1}^{2} M^i_k dx^k \tag{1.8b}$$

$$\partial_i \to \partial'_i = \sum_{k=1}^{2} [(M^t)^{-1}]^k_i \partial_k \tag{1.8c}$$

where M^i_j commutes with x^k, dx^k, and ∂_k, and M^t is the transposed matrix.

2. q-SEQUENCES, q-ANALYTIC FUNCTIONS, AND q-CURVES

If $f = f(\xi_1, \ldots, \xi_n)$ is a real function of n real variables ξ_m, which is analytic at the neighborhood of the origin, then it has the standard power series form

$$f = \sum_{N=0}^{\infty} \frac{1}{N!} \left(\sum_{m=1}^{N} \xi_m \frac{\partial}{\partial \xi_m} \right)^N f(0, \ldots, 0)$$

i.e.,

$$f = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(m_1 + \cdots + m_n)!}{m_1!m_2! \cdots m_n!} f_{m_1m_2\ldots m_n} \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_n^{m_n} \tag{2.1}$$

where the sum is over all nonnegative integers $(m_1, \ldots, m_n)$, and

$$f_{m_1\cdots m_n} = \frac{\partial^{(m_1+\cdots+m_n)}}{\partial\xi_1^{m_1} \cdots \partial\xi_n^{m_n}} f(0, \ldots, 0)$$

A q-sequence corresponding to the real analytic function f is defined as a set $\{a_{m_1m_2\cdots m_n}\}$ consisting of finite or infinite elements, in which there are the following structures:

(i) Some algebraic structures including the interior commutation relations within this q-sequence itself, and the combination of commutation relations between this q-sequence and other q-sequences, etc. These algebraic structures will be determined by some requirements, especially the so-called q-differential equations (Zhong, 1995a,b).

(ii) When $q \to 1$, the above algebraic relations must change into some relations among the real numbers $f_{m_1\cdots m_n}$, especially the algebraic relations which determine a power series solution of the corresponding differential equations. Now we take

$$\lim_{q\to 1} a_{m_1m_2\cdots m_n} = f_{m_1m_2\cdots m_n}$$

and if $(k_1, \ldots, k_n)$ are some n nonnegative integers such that $f_{k_1\cdots k_n} = 0$, then we take $a_{k_1\cdots k_n} = 0$ directly. In addition, two q-sequences are equal if and only if they are the same about (i) and (ii).

In this paper we only discuss the case of $n \leq 2$. For a q-sequence $\{a_{m_1m_2}\}$ we can write out a formal power series

$$f^q(x, y) = \sum_{m_1,m_2=0}^{\infty} \frac{[m_1 + m_2]_q!}{[m_1]_q![m_2]_q!} a_{m_1m_2} x^{m_1} y^{m_2}, \qquad (x, y) \in \mathscr{C}_2 \tag{2.2}$$

where the q-factorial is $[m]_q! = [m]_q[m-1]_q \cdots [1]_q$. We call f^q a q-analytic function on the quantum hyperplane $\mathscr{C}_2$. Evidently, the sum, the product, the multiplications by numbers, especially the quantum derivatives of arbitrary order determined by (1.4), etc., of q-analytic functions still are q-analytic functions.

In order to define a q-curve in the quantum hyperplane $\mathscr{C}_2$, we introduce two new q-analytic functions as follows: Let t be a real parameter and $\{a_n\}$, $\{b_m\}$ two q-sequences; then we write two q-analytic functions of t as

$$x(t) = \sum_{m=0}^{\infty} \frac{1}{[m]_q!} a_m t^m, \qquad y(t) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} b_n t^n \tag{2.3}$$

Let the commutation relation between $\{a_m\}$ and $\{b_n\}$ be

$$a_m b_n = q b_n a_m \qquad (m, n = 0, 1, \ldots) \tag{2.4}$$

Therefore we have

$$x(t)y(t) = qy(t)x(t) \tag{2.5}$$

The pair $(x(t), y(t))$ can be explained as a q-curve in the quantum hyperplane $\mathscr{C}_2$. Notice that when $t = 0$, then $x(0) = a_0$, $y(0) = b_0$, and $a_0b_0 = qb_0a_0$; therefore (a_0, b_0) is a "fixed point" in $\mathscr{C}_2$, and the above q-curve passes through it. As for the q-derivatives of these q-analytic functions, if we take the similar q-derivative, i.e.,

$$D^q(t^n) = \frac{(q^2t)^n - t^n}{q^2t - t} = [n]_q t^{n-1} \tag{2.6}$$

then we can obtain all derivatives of $x(t)$ and $y(t)$, e.g.,

$$\dot{x}(t) \equiv D^q x(t) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} a_{n+1} t^n \tag{2.7}$$

From equation (2.4), the commutation relation (2.5) still holds for the q-derivatives of arbitrary order of $x(t)$ and $y(t)$, in particular

$$\dot{x}(t)\dot{y}(t) = q\dot{y}(t)\dot{x}(t) \tag{2.8}$$

3. q-FIELDS AND q-SYMPLECTIC FORMS

Suppose that the commutation relation between two q-sequences $\{A_{mn}\}$ and $\{B_{kl}\}$ is (Zhong, 1993)

$$q^{-nk} A_{mn} B_{kl} = q^{1-m} B_{kl} A_{mn} \tag{3.1}$$

From the commutation relation (1.1), it is known that the commutation relation between the q-analytic functions

$$f^q(x, y) = \sum_{m,n=0}^{\infty} \frac{[m+n]_q!}{[m]_q![n]_q!} A_{mn} x^m y^n$$

$$g^q(x, y) = \sum_{k,l=0}^{\infty} \frac{[k+l]_q!}{[k]_q![l]_q!} B_{kl} x^k y^l$$

is

$$f^q g^q = q g^q f^q \tag{3.2}$$

f^q and g^q are called the components of the two-dimensional standard q-vector field $\mathbf{V}(x, y)$, which we denote by the brackets $\mathbf{V} = \{V^x, V^y\} = \{f^q(x, y), g^q(x, y)\}$. Here the so-called "components" concern some coordinate bases $(\mathbf{e}_x, \mathbf{e}_y)$. Let

$$B = \begin{bmatrix} e_1^1 & e_2^1 \\ e_1^2 & e_2^2 \end{bmatrix}$$

be a quantum matrix in $GL_q(2)$ as in (1.5); now we take $\mathbf{e}_x = (e_1^1, e_2^1)$ and $\mathbf{e}_y = (e_1^2, e_2^2)$ as the coordinate bases, and for these bases $\mathbf{V}$ is spanned as

$$\mathbf{V}(x, y) = V^x(x, y)\mathbf{e}_x + V^y(x, y)\mathbf{e}_y \tag{3.3}$$

In fact, here $\mathbf{V}$ corresponds to a q-analytic function pair $(\check{f}^q, \check{g}^q)$ obeying the commutation relation (1.1), since equation (3.3) can be written in Manin's (1988) form as

$$\begin{pmatrix} \check{f}^q \\ \check{g}^q \end{pmatrix} = \begin{pmatrix} e_1^1 & e_2^1 \\ e_1^2 & e_2^2 \end{pmatrix} \begin{pmatrix} f^q \\ g^q \end{pmatrix} \tag{3.4}$$

Generally, whether V^x, V^y satisfy (1.1) or not, $\mathbf{V} = V^x\mathbf{e}_x + V^y\mathbf{e}_y$ is called a q-vector field on $\mathscr{C}_2$; and if $V^xV^y = qV^yV^x$, then $\mathbf{V}$ is called a standard q-vector field on $\mathscr{C}_2$. From equations (2.3) and (2.5),

$$\mathbf{T}(t) = \{\dot{x}(t), \dot{y}(t)\} = \dot{x}(t)\mathbf{e}_x + \dot{y}(t)\mathbf{e}_y \tag{3.5}$$

is a standard q-vector corresponding to t. We call $\mathbf{T}(t)$ the tangent vector of the q-curve $(x(t), y(t))$.

If $M \in GL_q(2)$ is a q-matrix as in equation (1.5), then the basic vectors $\mathbf{e}'_x = (ae_1^1 + be_1^2, ae_2^1 + be_2^2)$ and $\mathbf{e}'_y = (ce_1^1 + de_1^2, ce_2^1 + de_2^2)$ correspond to the q-matrix $MB \in GL_{q^2}(2)$. It is easily seen that for this basic system, the components $\mathbf{V}$ are

$$V'^i = M_1^iV^x + M_2^iV^y \qquad (V'^i = V'^{x^i}) \tag{3.6}$$

Therefore $\mathbf{V}$ indeed has some vector properties.

Let $\tilde{\omega}^x$, $\tilde{\omega}^y$ denote the dual bases, i.e., we regard $\tilde{\omega}^i$ and $\mathbf{e}_j$ as two operators that act on each other such that

$$\tilde{\omega}^i(\mathbf{e}_j) = \delta_j^i = \mathbf{e}_j(\tilde{\omega}^i) \qquad (i, j = x, y) \tag{3.7}$$

If $\Omega_x(x, y)$ and $\Omega_y(x, y)$ are two q-analytic functions, $\tilde{\Omega}(x, y) = \tilde{\omega}^x\Omega_x(x, y) + \tilde{\omega}^y\Omega_y(x, y)$ is called a q-1-form field on $\mathscr{C}_2$. If and only if Ω_x and Ω_y satisfy a commutation relation as in (1.3e), i.e.,

$$\Omega_x\Omega_y = \frac{1}{q}\Omega_y\Omega_x \tag{3.8}$$

then $\tilde{\Omega}(x, y)$ is called a standard q-1-form field on the quantum hyperplane $\mathscr{C}_2$. The most interesting case for us is when this q-1-form field is generated by a q-analytic function and the q-differential calculus as in (1.2). For this reason, a q-analytic function

$$H^q(x, y) = \sum_{m,n=0}^{\infty} \frac{[m+n]_q!}{[m]_q![n]_q!} h_{mn}x^m y^n, \qquad (x, y) \in \mathscr{C}_2$$

is called a standard q-analytic function if and only if $\partial_x H^q$ and $\partial_y H^q$ are just the components of a standard q-1-form field on $\mathscr{C}_2$, i.e.,

$$\partial_x H^q \, \partial_y H^q = \frac{1}{q} \partial_y H^q \, \partial_x H^q \tag{3.9}$$

In particular, if we take $H^q = x$ and y, then we obtain $dx = \tilde{\omega}^x$ and $dy = \tilde{\omega}^y$, respectively. This means that if H^q is a standard q-analytic function, then dH^q is a standard q-1-form field on $\mathscr{C}_2$. From equation (3.9), H^q is a standard q-analytic function if and only if there are the following interior commutation relations in the q-sequence $\{h_{mn}\}$:

$$\sum_{\substack{m+k=R \\ n+l=S}}^{\infty} \frac{[m+n+1]_q!\,[k+l+1]_q!}{[m]_q!\,[n]_q!\,[k]_q!\,[l]_q!} q^{2n+k}(q^{-nk}h_{m+1,n}h_{k,l+1} - q^{-1-ml}h_{k,l+1}h_{m+1,n})$$

$$= 0, \qquad R, S = 0, 1, 2, \ldots \tag{3.10}$$

It is easily seen that the $GL_q(2)$ transformation rule of the components of a q-1-form field is

$$\Omega_i \to \Omega_i' = [(M^t)^{-1}]_i^x \Omega_x + [(M^t)^{-1}]_i^y \Omega_y, \qquad i = x, y \tag{3.11}$$

By the use of the actions of the operators defined in equation (3.7), we can define the action of a q-vector (q-1-form) field on a q-1-form (q-vector) field; the results are

$$\tilde{\Omega}(\mathbf{V}) = \Omega_x V^x + \Omega_y V^y \tag{3.12}$$

$$\mathbf{V}(\tilde{\Omega}) = V^x\Omega_x + V^y\Omega_y$$

Notice that this is different from the classical case in that $\tilde{\Omega}(\mathbf{V})$ and $\mathbf{V}(\tilde{\Omega})$ in general are not equal and may not necessarily be standard q-analytic functions unless $q \to 1$. However, they are invariants under the $GL_q(2)$ transformations.

Now we define the wedge product of two q-1-form fields by

$$\tilde{\omega}^i \wedge \tilde{\omega}^j = \tilde{\omega}^i \otimes \tilde{\omega}^j - \frac{1}{q} \check{R}^{ij}_{kl} \tilde{\omega}^k \otimes \tilde{\omega}^l \tag{3.13}$$

where $\check{R}$ is the Yang–Baxter matrix in (1.7). By the use of the Yang–Baxter relation (1.6), it is easily seen that equation (3.13) is covariant under the following $GL_q(2)$ transformation:

$$\tilde{\omega}^i \to \tilde{\omega}'^i = M^i_1 \tilde{\omega}^1 + M^i_2 \tilde{\omega}^2, \qquad \tilde{\omega}^i = \tilde{\omega}^{x^i} \tag{3.14}$$

and we have

$$\tilde{\omega}^x \wedge \tilde{\omega}^y = -\frac{1}{q} \tilde{\omega}^y \wedge \tilde{\omega}^x \tag{3.15}$$

Next, according to equation (1.2), the exterior differential of a q-1-form field $\tilde{\Omega} = \tilde{\omega}^x \Omega_x + \tilde{\omega}^y \Omega_y$ can be written as

$$d\tilde{\Omega} = \tilde{\omega}^x \wedge d\tilde{\Omega}_x + \tilde{\omega}^y \wedge d\tilde{\Omega}_y \tag{3.16}$$

From equation (1.7) the details of equation (3.13) are

$$\tilde{\omega}^x \wedge \tilde{\omega}^y = \frac{1}{q} \tilde{\omega}^x \otimes \tilde{\omega}^y - \frac{1}{q} \tilde{\omega}^y \otimes \tilde{\omega}^x \tag{3.17}$$

$$\tilde{\omega}^y \wedge \tilde{\omega}^x = \tilde{\omega}^y \otimes \tilde{\omega}^x - \frac{1}{q} \tilde{\omega}^x \otimes \tilde{\omega}^y$$

From the above, we have obtained the $GL_q(2)$-covariant q-symplectic forms; when $q \to 1$ they change into the symplectic forms on ordinary phase space.

4. NONCOMMUTATIVE HAMILTONIAN MECHANICS

Suppose that $H = H(x, p)$ is an ordinary Hamiltonian function on the ordinary phase space (x, p), where x is the generalized coordinate and p is the generalized momentum. H must be real analytic in the neighborhood of the origin. The corresponding q-analytic function is

$$H^q(x, p) = \sum_{m,n=0}^{\infty} \frac{[m+n]_q!}{[m]_q!\,[n]_q!} h_{mn} x^m p^n, \qquad x, p \in \mathscr{C}_2 \tag{4.1}$$

where we still use the notations x, p; however, (x, p) is a point in $\mathscr{C}_2$, $xp = qpx$. If and only if $q \to 1$ do x and p change into ordinary numbers. In particular, we require that H^q must be a standard q-analytic function, i.e., the q-sequence $\{h_{mn}\}$ obeys equation (3.10). This means that the interior commutation relations in $\{h_{mn}\}$ have been determined. If (k, l) is a nonnegative integer pair such that

$$\frac{\partial^{(k+l)}}{\partial x^k \partial p^l} H(0, 0) = 0$$

then we directly take $h_{kl} = 0$.

Now, we take the q-symplectic form on the q-phase hyperplane $\mathscr{C}_2$ as (notice the order)

$$\tilde{\Omega} = \tilde{\omega}^p \wedge \tilde{\omega}^x \tag{4.2}$$

where the wedge product is defined by equations (3.13) and (3.17). The orbit of a particle in the q-phase hyperplane $\mathscr{C}_2$ is defined as a q-curve $(x(t), p(t))$ in $\mathscr{C}_2$ (see Section 2, and let $y = p$), which must generate a Hamiltonian vector $\mathbf{U} = \{\dot{x}(t), \dot{p}(t)\}$ concerning the q-symplectic form $\tilde{\Omega}$ in equation (4.2), i.e., $\mathbf{U}$ satisfies the following equation:

$$\tilde{\Omega}(\cdot, \mathbf{U}) = dH^q \tag{4.3}$$

where d is the q-differential defined by equation (3.16). Therefore, from equation (3.7) we have

$$\begin{aligned} \dot{x}(t) &= \partial_p H^q(x, p) \\ \dot{p}(t) &= -\frac{1}{q}\, \partial_x H^q(x, p) \\ x(t) &= \sum_{m=0}^{\infty} \frac{1}{[m]_q!}\, a_m t^m \\ p(t) &= \sum_{n=0}^{\infty} \frac{1}{[n]_q!}\, b_n t^n \end{aligned} \tag{4.4}$$

This is the Hamiltonian equation in noncommutative mechanics, where ∂_i ($i = x, y$) is the quantum partial derivative, with the actions as in (1.4). Since $H^q(x, p)$ is a standard q-analytic function, the commutation relations (3.9) and (2.8) are consistent.

Now, from the ordinary Hamiltonian function $H(x, p)$ the real orbit $x = x(t)$ and $p = p(t)$ can be regarded as known, which means that the real numbers $\lim_{q\to 1} a_m$ and $\lim_{q\to 1} b_n$ are known. Therefore, to solve the q-differential equation (4.4), which is written in detail as

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{1}{[m]_q!}\, a_{m+1} t^m = \sum_{k,l=0}^{\infty} & \frac{[k + l + 1]_q!}{[k]_q!\, [l]_q!}\, q^k h_{k,l+1} \\ & \times \left(\sum_{r=0}^{\infty} \frac{1}{[r]_q!}\, a_r t^r \right)^k \left(\sum_{s=0}^{\infty} \frac{1}{[s]_q!}\, b_s t^s \right)^l \end{aligned} \tag{4.5}$$

$$\sum_{m=0}^{\infty} \frac{1}{[m]_q!} b_{m+1} t^m = \sum_{k,l=0}^{\infty} \frac{[k+l+1]_q!}{[k]_q!\,[l]_q!} q^{2l-1} h_{k+1,l} \times \left(\sum_{r=0}^{\infty} \frac{1}{[r]_q!} t^r\right)^k \left(\sum_{s=0}^{\infty} \frac{1}{[s]_q!} b_s t^s\right)^l$$

is equivalent to determining the combination of the commutation relations between the q-sequences $\{h_{mn}\}$ and $\{a_r\}$, $\{b_s\}$ from equation (4.5). We only need to contrast the coefficients of the terms t^M ($M = 0, 1, 2, \ldots$) on the right side of equation (4.5) with those of the left side; then the results can be directly written out. However, the general form is very tedious, and here there is no need to write out it.

The above method can be used for any one-dimensional single-particle system.

5. EXAMPLES

5.1. Free Particle with Mass M

In this case,

$$H(x, p) = \frac{p^2}{2M} \tag{5.1}$$

$$H^q(x, p) = \frac{1}{[2]_q!} h_{02} p^2$$

The q-Hamiltonian equation is

$$\sum_{m=0}^{\infty} \frac{1}{[m]_q!} a_{m+1} t^m = h_{02} \sum_{n=0}^{\infty} \frac{1}{[n]_q!} b_n t^n \tag{5.2}$$

$$\sum_{m=0}^{\infty} \frac{1}{[m]_q!} b_{m+1} t^m = 0$$

Therefore we have

$$a_0 = x_0, \qquad a_1 = h_{02} b_0 = h_{02} p_0, \qquad a_2 = a_3 = \cdots = 0 \tag{5.3}$$

$$b_0 = p_0, \qquad b_1 = b_2 = \cdots = 0$$

This means that the q-orbit is a noncommutative straight line passing through (x_0, p_0) in the quantum phase hyperplane $\mathscr{C}_2$,

$$x(t) = x_0 + h_{02} p_0 t, \qquad p(t) = p_0 \tag{5.4}$$

According to equations (2.4) and (5.3), we obtain the commutation relations

$$x_0 p_0 = q p_0 x_0, \qquad h_{02} p_0 = q p_0 h_{02} \tag{5.5}$$

Since $\lim_{q\to 1} h_{02} = 1/M$, h_{02} is related to some "noncommutative mass."

5.2. Noncommutative One-Dimensional Harmonic Oscillator

In this case, the classical Hamiltonian function is

$$H(x, p) = \frac{1}{2} M\theta x^2 + \frac{1}{2M} p^2 \tag{5.6}$$

where the frequency θ is a nonnegative real number. Then the q-Hamiltonian function is

$$H^q(x, p) = \frac{1}{[2]_q!} h_{20} x^2 + \frac{1}{[2]_q!} h_{02} p^2 \tag{5.7}$$

and the q-Hamiltonian equation is

$$\sum_{m=0}^{\infty} \frac{1}{[m]_q!} a_{m+1} t^m = h_{02} \sum_{n=0}^{\infty} \frac{1}{[n]_q!} b_n t^n \tag{5.8}$$

$$\sum_{m=0}^{\infty} \frac{1}{[m]_q!} b_{m+1} t^m = -\frac{1}{q} h_{20} \sum_{n=0}^{\infty} \frac{1}{[n]_q!} a_n t^n$$

In addition, from equation (3.10) we know that h_{02} must commute with h_{20}. Thus by equation (5.8) we obtain the following recurrence relations:

$$a_{m+1} = h_{02} b_m, \qquad a_{m+1} = -\frac{1}{q} h_{02} h_{20} a_{m-1} \tag{5.9}$$

$$b_{m+1} = -\frac{1}{q} h_{20} a_m, \qquad b_{m+1} = -\frac{1}{q} h_{02} h_{20} b_{m-1} \qquad (m = 1, 2, \ldots)$$

According to equation (2.4), we see that a natural supposition is that $h_{02}h_{20}$ commute with all a_m and b_n, and in this case we obtain the solution

$$x(t) = a_0 \cos_q(\sqrt{\alpha} t) + \frac{1}{\sqrt{\alpha}} h_{02} b_0 \operatorname{sin}q(\sqrt{\alpha} t) \tag{5.10}$$

$$y(t) = b_0 \cos_q(\sqrt{\alpha} t) - \frac{1}{(\sqrt{\alpha})^q} h_{20} a_0 \sin_q(\sqrt{\alpha} t)$$

where we have formally written $\alpha = (1/q) h_{02} h_{20}$, and the q-sine and q-cosine functions are defined by

$$\sin_q \zeta = \sum_{m=0}^{\infty} \frac{1}{[2m+1]_q!} (-1)^m \zeta^{2m+1} \tag{5.11}$$

$$\cos_q \zeta = \sum_{m=0}^{\infty} \frac{1}{[2m]_q!} (-1)^m \zeta^{2m}$$

ζ is a formal variable. The commutation relations are

$$\begin{aligned} a_0 b_0 &= q b_0 a_0 \\ h_{02} h_{20} &= h_{20} h_{02} \\ [h_{02} h_{20}, a_0 \text{ (or } b_0)] &= 0 \end{aligned} \tag{5.12}$$

From $\lim_{q \to 1} \alpha = \theta^2$, $\lim_{q \to 1} h_{02} = 1/M$, and $\lim_{q \to 1} h_{20} = M\theta^2$, therefore, the system given by (5.10) is a noncommutative one-dimensional harmonic oscillator; $\sqrt{\alpha}$ corresponds to a "frequency" and h_{02}, h_{20} are related to some "noncommutative mass."

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